Further improvements in Waring's Problem, III: Eighth powers.

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1. Introduction.

Let $G(k)$ denote the least number s such that every sufficiently large natural number is the sum of, at most, s kth powers of natural numbers. In this paper we develop certain technical extensions of the methods of [**6**], and thereby obtain the following improved bounds for $G(k)$ when $k = 8$.

Theorem. *We have* $G(8) \le 42$.

In [**6**] we obtained the bound $G(8) \le 43$.

Following [**3**], our methods are dependent on upper bounds for the number of solutions, $S_s^{(k)}(P,R)$, of the diophantine equations

$$x_1^k + \ldots + x_s^k = y_1^k + \ldots + y_s^k, \tag{1.1}$$

with $x_i, y_i \in \mathcal{A}(P,R)$, where throughout we write

$$\mathcal{A}(P,R) = \{1 \le n \le P \ : \ p \text{ prime}, p|n \text{ implies } p \le R\}.$$

In [**6**] we were preoccupied with refinements of the efficient differencing process, initiated in [**7**], and used to provide upper bounds for $S_s^{(k)}(P,R)$. Concerning the bound $G(8) \le 42$, the methods of [**6**] fail to reach a suitable bound for $S_s(P,P^\eta)$ by a power of P. The problem has, at its source, the inherent difficulty of estimating the contribution from the major arcs arising in the efficient differencing process. In this case, by obtaining improved estimates for the contribution of the major arcs arising in the latter part of the iteration process, we are able to reach the desired conclusion. Thus in §2 we develop a rather general method for this purpose, completing the proof of Theorem 1.1 in §4.

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Since the methods of this paper are so closely related to those of [**6**], we shall find it convenient to lift the notation used in the latter to this paper without comment. Additional pieces of notation will be made explicit as they become necessary.

2. Preliminary observations.

We first recall some of the notation of Vaughan and Wooley (1993). Throughout, k will denote an arbitrary integer exceeding 2, s will denote a positive integer, and ε and η will denote sufficiently small positive numbers. We take P to be a large positive real number depending at most on k, s, ε and η. We use $\ll$ and

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$\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on k, s, ε and η. We make frequent use of vector notation for brevity. For example, $(c_1, \ldots, c_t)$ is abbreviated to $\mathbf{c}$. Also, we shall write $e(\alpha)$ for $e^{2\pi i\alpha}$, and $[x]$ for the greatest integer not exceeding x.

In an effort to simplify our analysis, we adopt the following convention concerning the numbers ε and R. Whenever ε or R appear in a statement, either implicitly or explicitly, we assert that for each $\varepsilon > 0$, there exists a positive number $\eta_0(\varepsilon, s, k)$ such that the statement holds whenever $R = P^{\eta}$, with $0 < \eta \leq \eta_0(\varepsilon, s, k)$. Note that the "value" of ε, and η_0, may change from statement to statement, and hence also the dependency of implicit constants on ε and η. Since our iterative methods will involve only a finite number of statements (depending at most on k, s and ε), there is no danger of losing control of implicit constants through the successive changes implicit in our arguments. Finally, we use the symbol $\approx$ to indicate that constants and powers of R and P^{ε} are to be ignored.

For each $s \in \mathbb{N}$ we take $\phi_i = \phi_{i,s}$ $(i = 1, \ldots, k)$ to be real numbers, with $0 \leq \phi_i \leq 1/k$, to be chosen later. We then take

$$P_j = 2^j P, \quad M_j = P^{\phi_j}, \quad H_j = P_j M_j^{-k}, \quad Q_j = P_j (M_1 \ldots M_j)^{-1} \quad (1 \leq j \leq k).$$

We also write

$$\tilde{H}_j = \prod_{i=1}^{j} H_i \quad \text{and} \quad \tilde{M}_j = \prod_{i=1}^{j} M_i R \quad .$$

We define the modified forward difference operator, Δ_1^*, by

$$\Delta_1^* \left(f(x); h; m\right) = m^{-k} \left(f(x + hm^k) - f(x)\right),$$

and define Δ_j^* recursively by

$$\begin{aligned}\Delta_{j+1}^* &\left(f(x); h_1, \ldots, h_{j+1};\ m_1, \ldots, m_{j+1}\right) \\ &= \Delta_1^* \left(\Delta_j^* \left(f(x); h_1, \ldots, h_j; m_1, \ldots, m_j\right); h_{j+1}; m_{j+1}\right).\end{aligned}$$

We also adopt the convention that $\Delta_0^* \left(f(x); h; m\right) = f(x)$.

For $0 \leq j \leq k$ let

$$\Psi_j = \Psi_j(z; h_1, \ldots, h_j; m_1, \ldots, m_j) = \Delta_j^* \left(f(z); 2h_1, \ldots, 2h_j; m_1, \ldots, m_j\right)$$

where $f(z) = (z - h_1 m_1^k - \ldots - h_j m_j^k)^k$.

Write

$$f_j(\alpha) = \sum_{x \in \mathcal{A}(Q_j, R)} e(\alpha x^k) \quad \text{and} \quad g_j(\alpha) = \sum_{1 \leq x \leq Q_j} e(\alpha x^k).$$

Also, write

$$F_j(\alpha) = \sum_{z, \mathbf{h}, \mathbf{m}} e\left(\alpha \Psi_j(z; \mathbf{h}; \mathbf{m})\right),$$

where the summation is over $z, \mathbf{h}, \mathbf{m}$ with

$$1 \leq z \leq P_j, \quad M_i < m_i \leq M_i R, \quad m_i \in \mathcal{A}(P, R), \quad 1 \leq h_i \leq 2^{j-i} H_i \quad (1 \leq i \leq j). \tag{2.1}$$

We let $S_s^{(k)}(P, R)$ denote the number of solutions of the equation

$$x_1^k + \ldots + x_s^k = y_1^k + \ldots + y_s^k$$

with $x_i, y_i \in \mathcal{A}(P,R)$ $(1 \le i \le s)$. When no confusion is possible, we shall suppress the superscript k. Suppose that the real numbers λ_s and μ_s $(1 \le s < \infty)$ have the property that

$$S_s^{(k)}(P,R) \ll P^{\lambda_s+\varepsilon}. \tag{2.2}$$

Such numbers certainly exist, since we may trivially take $\lambda_s = 2s$ and $\mu_s = 2s$. Then, for each s, we define the quantity Δ_s by

$$\lambda_s = 2s - k + \Delta_s.$$

At the core of our argument is the use of a modified version of Vaughan and Wooley (1993; Lemma 2.2), which we now record.

Lemma 2.1. *Whenever $0 < t < s$ and $1 \le j \le k-1$, we have*

$$\int_0^1 |F_j(\alpha) f_j(\alpha)^{2s}| d\alpha \ll P^\varepsilon (Q_j^{\lambda_t})^{1/2} \left(\tilde{H}_j \tilde{M}_j M_{j+1}^{4s-2t-1} T_{j+1} \right)^{1/2},$$

where

$$T_{j+1} = P \tilde{H}_j \tilde{M}_{j+1} Q_{j+1}^{\lambda_{2s-t}} + \int_0^1 \left| F_{j+1}(\alpha) g_{j+1}(\alpha)^2 f_{j+1}(\alpha)^{4s-2t-2} \right| d\alpha.$$

Proof. The proof is almost identical to that of Vaughan and Wooley (1993; Lemma 2.2). We observe that in equation (3.5) of Wooley (1992), an upper bound for the quantity U_1 appearing in the proof of Lemma 3.1 of that paper is obtained by relaxing the restriction on x_1 and y_1, so that only $1 \le x_1, y_1 \le Q_{j+1}$. The lemma then follows as before, on considering the underlying diophantine equations.

Our argument will be based on a Hardy-Littlewood dissection, together with a suitable pruning operation. We now describe the various sets of arcs which we shall make use of.

Definition 2.2.

(i) Let $\mathfrak{m}_j$ denote the set of points in $[0,1]$ with the property that whenever there are $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q) = 1$, and

$$qP^{-1} Q_j^k R^{k(k-j)} |\alpha - a/q| \le 1, \tag{2.4}$$

then $q > P$. Further, let $\mathfrak{M}_j = [0,1] \setminus \mathfrak{m}_j$.

(ii) Let $\mathfrak{M}_j(q,a)$ be the set of α in $[0,1]$ for which (2.4) holds. (Note that the $\mathfrak{M}_j(q,a)$ with $0 \le a \le q \le P$ are disjoint.)

(iii) Let $\mathfrak{n}_j$ denote the set of points in $\mathfrak{M}_j$ with the property that whenever there are $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q) = 1$, and

$$q(PM_1)^{-\tau(k-j)} Q_j^k |\alpha - a/q| \le 1, \tag{2.5}$$

then $q > (PM_1)^{\tau(k-j)}$. Further, let $\mathfrak{N}_j = \mathfrak{M}_j \setminus \mathfrak{n}_j$.

(iv) Let $\mathfrak{N}_j(q,a)$ be the set of α in $\mathfrak{M}_j$ for which (2.5) holds. (Note that the $\mathfrak{N}_j(q,a)$ with $0 \le a \le q \le (PM_1)^{\tau(k-j)}$ are disjoint.)

Finally, we shall record the definitions of some generating functions of use on the various major arcs. We write

$$S(q,a) = \sum_{r=1}^{q} e(ar^k/q), \quad \text{and} \quad v_j(\beta) = \sum_{1 \le x \le Q_j^k} \frac{1}{k} x^{1/k-1} e(\beta x),$$

and

$$V(\alpha; q, a) = q^{-1} S(q, a) v(\alpha - a/q),$$

We then define $g_j^*(\alpha)$ to be the function of α taking the value zero whenever $\alpha \in \mathfrak{n}_j$, and by $g_j^*(\alpha) = V(\alpha; q, a)$ whenever $\alpha \in \mathfrak{N}_j(q, a)$ and $0 \le a \le q \le (PM_1)^{\tau(k-j)}$. Also, we write

$$\tau_j(q, a, \mathbf{h}, \mathbf{m}) = \left| \sum_{r=1}^{q} e\left(\frac{a}{q} \Psi_j(r, \mathbf{h}, \mathbf{m}) \right) \right|.$$

We then define $F_j^*(\alpha)$ to be the function of α taking the value zero whenever $\alpha \in \mathfrak{m}_j$, and by

$$F_j^*(\alpha) = \sum_{\mathbf{m}} \sum_{\mathbf{h}} \frac{P q^{-1} \tau_j(q, a, \mathbf{h}, \mathbf{m})}{(1 + |\alpha - a/q| h_1 \dots h_j P^{k-j})^{\frac{1}{k-j}}}$$

whenever $\alpha \in \mathfrak{M}_j(q, a)$ and $0 \le a \le q \le P$.

3. A refined Hardy-Littlewood dissection for larger k.

Our first step towards a new major arc estimate is a pruning operation, which we perform in the following lemma.

Lemma 3.1. *Suppose that $1 \le j \le k-4$, and write $\tau = 2^{1+j-k}$. Let u be a positive integer, and define*

$$t = \left[\left(\frac{k-j+1}{k-j} \right) u + 1 \right], \quad \theta = t - \left(\frac{k-j+1}{k-j} \right) u,$$

and

$$\nu_u = \frac{k-j}{k-j+1} (\theta \Delta_{t-1} + (1-\theta) \Delta_t).$$

Then

$$\int_0^1 |F_j(\alpha) g_j(\alpha)^2 f_j(\alpha)^{2u}| d\alpha \ll P^{1+\varepsilon} \tilde{H}_j \tilde{M}_j Q_j^{2u+2-k} \mathcal{M} + I_{j,u},$$

where

$$\mathcal{M} = (PM_1)^{-\tau} Q_j^{\Delta_{u+1}} + (PM_1)^{(k-j)\tau} Q_j^{\nu_u - 2},$$

and

$$I_{j,u} = \int_{\mathfrak{N}_j} |F_j^*(\alpha) g_j^*(\alpha)^2 f_j(\alpha)^{2u}| d\alpha.$$

Proof. In view of Definition 2.2, we may imitate the analysis of the proof of Vaughan and Wooley (1993; Lemma 13.1) to deduce that

$$\int_0^1 |F_j(\alpha) g_j(\alpha)^2 f_j(\alpha)^{2u}| d\alpha \ll I_1 + I_2, \tag{3.1}$$

where

$$I_1 = \int_{\mathfrak{M}_j} |F_j^*(\alpha) g_j(\alpha)^2 f_j(\alpha)^{2u}| d\alpha, \tag{3.2}$$

and

$$I_2 = \left(P^{\frac{k-j-1}{k-j}+\varepsilon}\tilde{H}_j\tilde{M}_j + \sup_{\alpha\in\mathfrak{m}_j}|F_j(\alpha)|\right)\int_0^1 |g_j(\alpha)^2 f_j(\alpha)^{2u}|d\alpha. \tag{3.3}$$

By using a Weyl differencing argument, we may follow the pattern established in Vaughan and Wooley (1993; Lemmata 6.1 and 12.1) to deduce, from Lemma 4.1 and Corollary 4.2.1 of that paper, that

$$\sup_{\alpha\in\mathfrak{m}_j}|F_j(\alpha)| \ll P^{1+\varepsilon}\tilde{H}_j\tilde{M}_j(PM_1)^{-\tau}.$$

On noting the observation at the end of section 3 of Wooley (1992), we have also the bound

$$\int_0^1 |g_j(\alpha)^2 f_j(\alpha)^{2u}|d\alpha \ll Q_j^{\lambda_{u+1}+\varepsilon}.$$

Also, since

$$\frac{2^{k-j-1}}{k-j} \geq 2 > 1+\phi_1$$

when $k-j\geq 4$, we have

$$P^{\frac{k-j-1}{k-j}}\tilde{H}_j\tilde{M}_j \ll P\tilde{H}_j\tilde{M}_j(PM_1)^{-\tau}.$$

Thus

$$I_2 \ll P^{1+\varepsilon}\tilde{H}_j\tilde{M}_j Q_j^{2u+2-k}(PM_1)^{-\tau}Q_j^{\Delta_{u+1}}. \tag{9.3}$$

Suppose now that $\alpha\in\mathfrak{n}_j$. By Dirichlet's Theorem we can choose a and q with $(a,q)=1$, $q\leq (PM_1)^{-\tau(k-j)}Q_j^k$ and $|q\alpha-a|\leq (PM_1)^{\tau(k-j)}Q_j^{-k}$. Then since $\alpha\notin\mathfrak{N}$, we have

$$q+Q_j^k|q\alpha-a| > (PM_1)^{\tau(k-j)}.$$

By Definition 4.9(iii) of FIWP together with Lemma 4.8 of FIWP, we therefore have

$$\sup_{\alpha\in\mathfrak{n}_j\cap\mathfrak{M}_j(q,a)}|F_j^*(\alpha)| \ll \sum_{\mathbf{m}}\sum_{\mathbf{h}} P(q+Q_j^k|q\alpha-a|)^{-\frac{1}{k-j}} \ll P\tilde{H}_j\tilde{M}_j(PM_1)^{-\tau}.$$

Then as above we deduce that

$$\int_{\mathfrak{n}_j}|F_j^*(\alpha)g_j(\alpha)^2f_j(\alpha)^{2u}|d\alpha \ll P^{1+\varepsilon}\tilde{H}_j\tilde{M}_jQ_j^{2u+2-k}(PM_1)^{-\tau}Q_j^{\Delta_{u+1}}. \tag{3.5}$$

Finally, we consider

$$I_3 = \int_{\mathfrak{N}_j}|F_j^*(\alpha)g_j(\alpha)^2f_j(\alpha)^{2u}|d\alpha. \tag{3.6}$$

By Vaughan (1981b; Theorem 4.1), when $\alpha\in\mathfrak{N}_j(q,a)$ we have

$$\begin{aligned} g_j(\alpha)-g_j^*(\alpha) &\ll \big(q+Q_j^k|q\alpha-a|\big)^{\frac{1}{2}+\varepsilon} \\ &\ll (PM_1)^{\frac{1}{2}\tau(k-j)+\varepsilon}. \end{aligned}$$

Thus

$$I_3 \ll (PM_1)^{\tau(k-j)+\varepsilon}\int_{\mathfrak{N}_j}|F_j^*(\alpha)f_j(\alpha)^{2u}|d\alpha + \int_{\mathfrak{N}_j}|F_j^*(\alpha)g_j^*(\alpha)^2f_j(\alpha)^{2u}|d\alpha. \tag{3.7}$$

The lemma now follows from (3.6), (3.7), (3.5) and (3.4) provided that we can obtain a suitable estimate for

$$J = \int_{\mathfrak{N}_j} |F_j^*(\alpha) f_j(\alpha)^{2u}| d\alpha.$$

By Hölder's inequality,

$$J^{k-j+1} \ll \left(\int_0^1 |F_j^*(\alpha)|^{k-j+1} d\alpha \right) \left(\int_0^1 |f_j(\alpha)|^{2t-2} d\alpha \right)^{(k-j)\theta} \left(\int_0^1 |f_j(\alpha)|^{2t} d\alpha \right)^{(k-j)(1-\theta)},$$

and so by Vaughan and Wooley (1993; Lemma 4.10),

$$J \ll P^{1+\varepsilon} \tilde{H}_j \tilde{M}_j Q_j^{2u-k+\nu_u}.$$

There are a number of approaches to the task of estimating the integral $I_{j,u}$ appearing in the statement of Lemma 3.1, the effectiveness of the respective methods depending on the relative sizes of Q_j and $(PM_1)^{\tau(k-j)}$. Although we shall require only a relatively simple approach in our applications, we explain two ideas, so as to make our general approach more complete.

Lemma 3.2. *Suppose that $j \le k-4$. Let u be a positive integer, and define*

$$\gamma = 1 - \frac{1}{k-j+1} - \frac{2}{k+1}, \quad t = \left[\gamma^{-1} u + 1 \right], \quad \theta = t - \gamma^{-1} u,$$

and

$$\rho_u = \gamma \left(\theta \Delta_{t-1} + (1-\theta) \Delta_t \right).$$

Then

$$I_{j,u} \ll P^{1+\varepsilon} \tilde{H}_j \tilde{M}_j Q_j^{2u+2-k+\rho_u}.$$

Proof. By Hölder's inequality, we have

$$I_{j,u} \ll J_1^{\frac{1}{k-j+1}} J_2^{\frac{2}{k+1}} \left(\int_0^1 |f_j(\alpha)|^{2t-2} d\alpha \right)^{\gamma\theta} \left(\int_0^1 |f_j(\alpha)|^{2t} d\alpha \right)^{\gamma(1-\theta)},$$

where

$$J_1 = \int_{\mathfrak{N}_j} |F_j^*(\alpha)|^{k-j+1} d\alpha,$$

and

$$J_2 = \int_{\mathfrak{N}_j} |g_j^*(\alpha)|^{k+1} d\alpha.$$

But by Vaughan and Wooley (1993; Lemma 4.10), we have $J_1 \ll P^{\varepsilon} (P\tilde{H}_j \tilde{M}_j)^{k-j+1} Q_j^{-k}$. Further, by the methods of Vaughan (1981b; section 4.4), we have $J_2 \ll Q_j^{1+\varepsilon}$. Thus

$$I_{j,u} \ll P^{1+\varepsilon} \tilde{H}_j \tilde{M}_j Q_j^{2-k} \left(Q_j^{\lambda_{t-1}+k} \right)^{\gamma\theta} \left(Q_j^{\lambda_t + k} \right)^{\gamma(1-\theta)},$$

and the lemma now follows.

Lemma 9.3. *Suppose that $j \le k-4$. Let u be a positive integer, and define*

$$\gamma = 1 - \frac{1}{k-j+1} - \frac{2}{k+1}, \quad t = \left[\gamma^{-1}(u-\tfrac{1}{2})+1\right], \quad \theta = t - \gamma^{-1}(u-\tfrac{1}{2}),$$

and

$$\sigma_u = \gamma\left(\theta\Delta_{t-1} + (1-\theta)\Delta_t\right).$$

Suppose also that $u \ge 2\gamma k$, and $(PM_1)^{\tau(k-j)} \ll Q_j^{2/3}$. Then

$$I_{j,u} \ll P^{1+\varepsilon}\tilde{H}_j\tilde{M}_jQ_j^{2u+2-k}\left((PM_1)^{\tau(k-j)/8}Q_j^{\sigma_u-1/4}+1\right).$$

Proof. Let $M = Q_j^{\frac{1}{2}}(PM_1)^{\tau(k-j)/4}$. Suppose that $\alpha \in \mathfrak{N}_j$. Then by Vaughan and Wooley (1991; Lemma 7.2), we have

$$f_j(\alpha) \ll P^{\varepsilon}(f_j^*(\alpha) + Q_j^{3/4}(PM_1)^{\tau(k-j)/8}),$$

where

$$f_j^*(\alpha) = Q_j(q + Q_j^k|\alpha q - a|)^{-\frac{1}{2k}}$$

when $\alpha \in \mathfrak{N}_j(q,a)$, and is zero otherwise. Therefore

$$I_{j,u} \ll P^{\varepsilon}(K_1+K_2), \tag{3.8}$$

where

$$K_1 = \int_{\mathfrak{N}_j} |F_j^*(\alpha)g_j^*(\alpha)^2f_j^*(\alpha)f_j(\alpha)^{2u-1}|d\alpha$$

and

$$K_2 = Q_j^{3/4}(PM_1)^{\tau(k-j)/8}\int_{\mathfrak{N}_j} |F_j^*(\alpha)g_j^*(\alpha)^2f_j(\alpha)^{2u-1}|d\alpha.$$

By Hölder's inequality, we have

$$K_2 \ll Q_j^{3/4}(PM_1)^{\tau(k-j)/8}L_1^{\frac{1}{k-j+1}}L_2^{\frac{2}{k+1}}L_3^{\gamma\theta}L_4^{\gamma(1-\theta)},$$

where

$$L_1 = \int_{\mathfrak{N}_j} |F_j^*(\alpha)|^{k-j+1}d\alpha,$$

$$L_2 = \int_{\mathfrak{N}_j} |g_j^*(\alpha)|^{k+1}d\alpha,$$

$$L_3 = \int_0^1 |f_j(\alpha)|^{2t-2}d\alpha,$$

$$L_4 = \int_0^1 |f_j(\alpha)|^{2t}d\alpha.$$

Then by Vaughan and Wooley (1993; Lemma 4.10) and the methods of Vaughan (1981b; section 4.4), we have

$$K_2 \ll P^{1+\varepsilon}\tilde{H}_j\tilde{M}_jQ_j^{2u+2-k}Q_j^{\sigma_u-1/4}(PM_1)^{\tau(k-j)/8}.$$

If K_2 is the dominating contribution to the right hand side of (3.8) then we are done. Otherwise,

$$I_{j,u} \ll P^{\varepsilon} K_1,$$

and by Hölder's inequality,

$$K_1 \ll K_3^{\frac{1}{2u}} I_{j,u}^{1-\frac{1}{2u}},$$

where

$$K_3 = \int_{\mathfrak{N}_j} |F_j^*(\alpha) g_j^*(\alpha)^2 f_j^*(\alpha)^{2u}| d\alpha.$$

Then

$$K_1 \ll K_3^{\frac{1}{2u}} \left(P^{\varepsilon} K_1\right)^{1-\frac{1}{2u}},$$

and hence $K_1 \ll P^{\varepsilon} K_3$. Next, by Hölder's inequality,

$$K_3 \ll L_1^{\frac{1}{k-j+1}} L_2^{\frac{2}{k+1}} K_4^{\gamma},$$

where

$$\begin{aligned} K_4 &= \int_{\mathfrak{N}_j} |f_j^*(\alpha)|^{2u/\gamma} d\alpha \\ &= Q_j^{2u/\gamma} \sum_{q \le (PM_1)^{\tau(k-j)/8}} \sum_{\substack{1 \le a \le q \\ (a,q)=1}} \int_{\mathfrak{N}_j(q,a)} \left(q + Q_j^k |\alpha q - a|\right)^{-\frac{u}{\gamma k}} d\alpha. \end{aligned}$$

Then provided that $u \ge 2\gamma k$, we may deduce that $K_4 \ll P^{\varepsilon} Q_j^{\frac{2u}{\gamma} - k}$, and the desired conclusion follows.

Let us now briefly explore some of the consequences of the above treatments. Our aim is to show that

$$\int_0^1 \left|F_j(\alpha) g_j(\alpha)^2 f_j(\alpha)^{2u}\right| d\alpha \ll P^{1+\varepsilon} \tilde{H}_j \tilde{M}_j Q_j^{2u+2-k} \left((PM_1)^{-\tau} Q_j^{\Delta_{u+1}} + 1\right).$$

Suppose first that

$$\Delta_{u+1}(1 - \phi_1 - \ldots - \phi_j) - \tau(1 + \phi_1) > 0.$$

Then from Lemma 9.1, 9.2, and (9.3), the above conclusion holds whenever

$$(k - j + 1)\tau(1 + \phi_1) \le (2 + \Delta_{u+1} - \nu_u)(1 - \phi_1 - \ldots - \phi_j),$$

and either

(i) $(\Delta_{u+1} - \rho_u)(1 - \phi_1 - \ldots - \phi_j) \ge \tau(1 + \phi_1)$,

or

(ii) $u \ge 2\gamma k$, $\tau(k - j)(1 + \phi_1) \ge \frac{2}{3}(1 - \phi_1 - \ldots - \phi_j)$, and

$$(\Delta_{u+1} - \sigma_u + \frac{1}{4})(1 - \phi_1 - \ldots - \phi_j) \ge \frac{9}{8}\tau(k - j).$$

Alternatively, if

$$\Delta_{u+1}(1 - \phi_1 - \ldots - \phi_j) - \tau(1 + \phi_1) \le 0,$$

then the desired conclusion holds whenever

$$\tau(k - j)(1 + \phi_1) + (\nu_u - 2)(1 - \phi_1 - \ldots - \phi_j) \le 0,$$

and either

(i) $\rho_u \le 0$, or

(ii) $u \ge 2\gamma k$, $\tau(k-j)(1+\phi_1) \ge \frac{2}{3}(1-\phi_1-\ldots-\phi_j)$, and

$$\frac{1}{8}\tau(k-j)(1+\phi_1) + (\sigma_u - \frac{1}{4})(1-\phi_1-\ldots-\phi_j) \le 0.$$

4. The iterative procedure for eighth powers.

We now turn our attention to the main task of developing the iterative procedures for eighth powers. Following the notation of FIWP, our iterative procedures will be based on schemes of the following form.

$$\begin{array}{ccccccccc} F_0^2 f_0^{2s-2} & \longmapsto F_1 f_1^{2s-2} & \longrightarrow & F_2 f_2^{2s-2} & \longrightarrow & \ldots & \longrightarrow & F_j f_j^{2s-2} & \Longrightarrow (F_j)(f_j^{2s-2}) \\ & & & \big\downarrow & & & & \big\downarrow & \\ & & & f_1^{2s-2} & & & & f_{j-1}^{2s-2} & \end{array}$$

Suppose that condition (I) of §9 holds. Then λ_s and $\underline{\phi}$ are determined by the equations

$$P\tilde{H}_{j-1}\tilde{M}_j Q_j^{\lambda_{s-1}} \approx P\tilde{H}_j\tilde{M}_j Q_j^{\lambda_{s-1}}(PM_1)^{-\tau}, \tag{10.1}$$

$$P\tilde{H}_{i-1}\tilde{M}_{i-1}Q_i^{\lambda_{s-i}} \approx \Big(P(\tilde{M}_i\tilde{H}_i)^2 M_{i+1}^{2s-2} Q_{i+1}^{\lambda_{s-1}} Q_i^{\lambda_{s-1}}\Big)^{1/2} \qquad (1 \le i \le j), \tag{10.2}$$

$$P^{\lambda_s} \approx PM_1^{2s-2}Q_1^{\lambda_{s-1}}. \tag{10.3}$$

Then following through the analysis of §13 of FIWP, we find that

$$\phi_1 = \frac{\frac{1}{k+\Delta} + \left(\frac{1-\tau}{k} - \frac{1}{k+\Delta}\right)\alpha^{j-1}}{1 + \frac{\tau}{k}\alpha^{j-1}},$$

where $\alpha = \frac{k-\Delta}{2k}$ and $\Delta = \Delta_{s-1}$. Further, λ_s^* is given by

$$\lambda_s^* = \lambda_{s-1}^*(1-\phi_1) + 1 + (2s-2)\phi_1. \tag{10.3a}$$

Meanwhile, if conditions (II) of §9 hold, then λ_s and $\underline{\phi}$ are determined by the equations (10.2), (10.3) and

$$P\tilde{H}_{j-1}\tilde{M}_j Q_j^{\lambda_{s-1}} \approx P\tilde{H}_j\tilde{M}_j Q_j^{2s-2-k}. \tag{10.4}$$

Equations (10.3) and (10.4) lead to the equations

$$k\phi_j = 1 - \Delta(1-\phi_1-\ldots-\phi_j),$$
$$2k\phi_i = 1 + (k-\Delta)\phi_{i+1} \quad (1 \le i \le j).$$

Write $\alpha = \frac{k-\Delta}{2k}$, and define a_i, b_i and c_i $(1 \le i \le j)$ by

$$a_j = 1-\Delta, \quad b_j = \Delta, \quad c_j = k - \Delta,$$

and for $1 \le i < j$ by

$$a_i = 1 + (k - \Delta)a_{i+1}c_{i+1}^{-1}, \quad b_i = (k - \Delta)b_{i+1}c_{i+1}^{-1}, \quad c_i = 2k - (k - \Delta)b_{i+1}c_{i+1}^{-1}.$$

Then we deduce that

$$\phi_i = c_i^{-1}\left(a_i + b_i(\phi_1 + \ldots + \phi_{i-1}\right) \quad (1 < i \le j),$$

and

$$\phi_1 = a_1/c_1.$$

We then find that λ_s is again determined by (10.3a).

We now turn our attention to the case $k = 8$, and divide into cases according to the value of s.

5. The treatment for eighth powers.

Henceforth we put $k = 8$. We divide into cases according to the value of s. Our analysis will be made a little simpler by noting that Δ_s may be taken to be zero when s is sufficiently large.

Lemma 5.1. *We may take $\lambda_s = 2s - 8$ when $s \ge 22$.*

Proof. By reference to the Appendix of FIWP, when $k = 8$ we have $\lambda_{16}^* \le 24.1954446$ and $\lambda_{18}^* \le 28.0945483$. Let $\mathfrak{m}$ denote the set of real numbers α with the property that whenever $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a, q) = 1$ and $|q\alpha - a| \le P^{-7}$, then one has $q > P$. Further, define

$$f(\alpha) = \sum_{x \in \mathcal{A}(P,R)} e(\alpha x^k) \quad \text{and} \quad g(\alpha) = \sum_{1 \le x \le P} e(\alpha x^k).$$

Then by the argument of the proof of Vaughan (1989; Theorem 1.8), we have

$$\sup_{\alpha \in \mathfrak{m}} |f(\alpha)| \ll P^{1-\sigma+\varepsilon},$$

where on recalling (2.2),

$$\sigma = \frac{1 - \Delta_{16}}{64} \ge 0.0125711.$$

We now consider the mean value

$$\int_0^1 |g(\alpha)^2 f(\alpha)^{2s-2}| d\alpha$$

when $s \ge 22$, which, on considering the underlying diophantine equation, plainly provides an upper bound for $S_s(P, R)$. The contribution from the minor arcs $\mathfrak{m}$ is at most

$$\left(\sup_{\alpha \in \mathfrak{m}} |f(\alpha)|\right)^{2s-36} \int_0^1 |g(\alpha)^2 f(\alpha)^{34}| d\alpha \ll P^{(2s-36)(1-\sigma)} P^{\lambda_{18}^* + \varepsilon}.$$

A little calculation reveals that $(2s - 36)(1 - \sigma) + \lambda_{18}^* < 2s - 8$ whenever $s \ge 22$, and hence the minor arc contribution is acceptable. Meanwhile, the major arc contribution can be estimated satisfactorily via a standard pruning argument, owing to the presence of the classical Weyl sums $g(\alpha)$. We may therefore conclude that

$$\int_0^1 |g(\alpha)^2 f(\alpha)^{2s-2}| d\alpha \ll P^{2s-8},$$

and the lemma follows.

We now consider successively the values of s in the range $15 \leq s \leq 19$.

(a) $s = 15$.
We use the Corollary to Lemma 4.1 with $j = 4$ and $u = 13$. First note that by Lemma 5.1, we have

$$\rho_{13} = \tfrac{13}{45}(\Delta_{22} + \Delta_{23}) = 0.$$

Next, by reference to the Appendix of FIWP, when $k = 8$ we may take $\lambda_{14} = 20.3659701$. Consequently

$$\Delta_{14} = 0.3659701 > 0.28125 = \frac{w(k+1)}{k-j}.$$

Then (4.2) holds. Also,

$$2 + \Delta_{14} - \nu_{13} \geq 2 > 1.40625 = \frac{w(k+1)(k-j+1)}{k-j},$$

and so (4.3) holds. Then by the Corollary to Lemma 4.1, it follows that λ_{15}^* is given by (4.10) with ϕ_1 given by (4.7). Thus we obtain $\phi_1 \leq 0.11822800$ and $\lambda_{15}^* \leq 22.2685262$.

(b) $s = 16$.
We use Lemma 4.1 with $j = 4$ and $u = 14$. As in the case $s = 15$, we find that $\rho_{14} = 0$, and that condition (A) (which follows from (4.3)) is satisfied easily. Then provided only that (α) holds, that is

$$\Delta_{15}(1 - \phi_1 - \ldots - \phi_4) \geq \tfrac{1}{8}(1 + \phi_1),$$

we may deduce that the ϕ_i are given by (4.7), (4.8) and (4.9). A calculation reveals that the latter equations give $\phi_1 \leq 0.11942505$, $\phi_2 \leq 0.11780429$, $\phi_3 \leq 0.11445019$, and $\phi_4 \leq 0.10750899$. Thus the desired condition is indeed met, and by (4.10) we have $\lambda_{16}^* \leq 24.1918579$.

(c) $s = 17$.
We use the Corollary to Lemma 4.1 with $j = 3$ and $u = 15$. We find once again that $\rho_{15} = 0$, and that (4.3) is satisfied easily. Condition (4.2) is also satisfied, since by using the conclusion of part (b), we have

$$\Delta_{16} = 0.1918579 > 0.1125 = \frac{w(k+1)}{k-j}.$$

Hence we may deduce that the ϕ_i are given by (4.7), (4.8) and (4.9). A calculation reveals that $\phi_1 \leq 0.12068453$, and hence by (4.10), $\lambda_{17}^* \leq 26.1341799$.

(d) $s = 18$.
We use the Corollary to Lemma 4.1 with $j = 3$ and $u = 16$. Once more, we find that $\rho_{16} = 0$, and we have

$$\Delta_{17} = 0.1341799 > 0.1125 = \frac{w(k+1)}{k-j}.$$

Thus conditions (4.2) and (4.3) are satisfied easily. The ϕ_i are therefore given by (4.7), (4.8) and (4.9). Then $\phi_1 \leq 0.12131915$, and hence by (4.10), $\lambda_{18}^* \leq 28.0884545$.
(e) $s = 19$.
We use the Corollary to Lemma 4.1 with $j = 2$ and $u = 17$. Again we find that $\rho_{17} = 0$. Also

$$\Delta_{18} = 0.0884545 > 0.046875 \geq \frac{w(k+1)}{k-j}.$$

Hence conditions (4.2) and (4.3) are met, and we may deduce that the ϕ_i are given by (4.7), (4.8) and (4.9). Then we have $\phi_1 \leq 0.12214150$, and hence by (4.10), $\lambda_{19}^* \leq 30.0547826$.

We now complete the proof of the theorem. Let $X = P^{\frac{8}{15}}$ and $Z = PX^{-1}$. Define the generating function $h(\alpha)$ by

$$h(\alpha) = \sum_{x \in \mathcal{C}} e(\alpha x^8),$$

where

$$\mathcal{C} = \{x \ : \ x = pz, \ X/2 < p \leq X, \ p \text{ prime}, \ z \in \mathcal{A}(Z, Z^\eta)\}.$$

Let s be an even integer, and write $s = 2r$. Define $\mathfrak{m}$ to be the set of real numbers α in $(\frac{1}{16}P^{-7}, 1 + \frac{1}{16}P^{-7}]$ with the property that whenever $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a,q) = 1$ and $|\alpha - a/q| \leq q^{-1}X^{-7}(rZ^8)^{-1}$, then one has $q > X$. Then the argument of Vaughan (1989; section 9) gives

$$\sup_{\alpha \in \mathfrak{m}} |h(\alpha)| \ll P^{1-\sigma+\varepsilon}, \tag{5.1}$$

where

$$\sigma = \frac{8 - 7\Delta_s}{30s}. \tag{5.2}$$

By (5.2) with $s = 16$, we obtain $\sigma > 0.01386873$. Moreover $\lambda_{19}^* + 4(1-\sigma) < 34$. Then by Vaughan and Wooley (1991; Theorem 4), we may finally conclude that $G(8) \leq 42$.

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